Bayesian inference for the Gaussian

* latent mean, fixed variance

$\underline{x} = \{ x_1, x_2, \dots, x_N \}$

$$p(\underline{x}, \mu) = p(\underline{x} \mid \mu)\, p(\mu)$$

$$\rightarrow p(\underline{x} \mid \mu) = \prod_i p(x_i \mid \mu)$$

$$= \prod_i \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[- \frac{(x_i - \mu)^2}{2\sigma^2} \right]$$

$$= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[- \frac{1}{2\sigma^2} \sum_{i=1}^{N} (x_i - \mu)^2 \right]$$

$$\rightarrow p(\mu) = \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(- \frac{(\mu - \mu_0)^2}{2\sigma_0^2} \right)$$

$$p(\mu \mid \underline{x}) \propto p(\underline{x} \mid \mu)\, p(\mu).$$

$$\propto \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{N} (x_i - \mu)^2 - \frac{1}{2\sigma_0^2} (\mu - \mu_0)^2 \right]$$

$$\propto \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{N} \left(x_i^2 + \mu^2 - 2\mu x_i \right) - \frac{1}{2\sigma_0^2} \left(\mu^2 + \mu_0^2 - \cancel{2\mu\mu_0} \right) \right]$$

$$\propto \exp\left[-\frac{N}{2\sigma^2}\mu^2 + \frac{\mu}{\sigma^2} \sum_{i=1} x_i - \frac{\mu^2}{2\sigma_0^2} + \frac{\mu_0}{\sigma_0^2}\mu \right]$$

$$\propto \exp\left[-\mu^2 \left(\frac{N}{2\sigma^2} + \frac{1}{2\sigma_0^2} \right) + \mu \left(\frac{\mu_0}{\sigma_0^2} + \frac{1}{\sigma^2} \sum_i x_i \right) \right]$$

Forme quadratique ds l'exp. donc Gaussienne :

$$p(\mu \mid \underline{x}) \propto \exp\left(\frac{-(\mu - \mu_*)^2}{2\sigma_*^2} \right) \propto \exp\left[-\mu^2 \frac{1}{2\sigma_*^2} + \mu \frac{\mu_*}{\sigma_*^2} \right]$$

On procède par identification.

$$\Rightarrow \frac{1}{2\sigma_*^2} = \frac{N}{2\sigma^2} + \frac{1}{2\sigma_0^2}$$

$$\Leftrightarrow \boxed{\frac{1}{\sigma_*^2} = \frac{N}{\sigma^2} + \frac{1}{\sigma_0^2}}$$

$$\rightarrow \frac{\mu_*}{\sigma_*^2} = \frac{\mu_0}{\sigma_0^2} + \frac{1}{\sigma^2} \sum_i x_i$$

$$\Leftrightarrow \mu^* = \sigma_*^2 \left(\frac{\mu_0}{\sigma_0^2} + \frac{\sum_i x_i}{\sigma^2} \right)$$

$$= \left(\frac{N}{\sigma^2} + \frac{1}{\sigma_0^2} \right)^{-1} \left(\frac{\mu_0}{\sigma_0^2} + \frac{\sum_i x_i}{\sigma^2} \right).$$

$$= \frac{\sigma^2 \times \sigma_0^2}{N\sigma_0^2 + \sigma^2} \left(\frac{\mu_0}{\sigma_0^2} + \frac{\sum_i x_i}{\sigma^2} \right)$$

$$= \frac{\sigma^2}{N\sigma_0^2 + \sigma^2} \mu_0 + \frac{\sigma_0^2}{N\sigma_0^2 + \sigma^2} \sum_i x_i$$

$$\boxed{\mu_* = \frac{\sigma^2}{N\sigma_0^2 + \sigma^2} \mu_0 + \frac{N\sigma_0^2}{N\sigma_0^2 + \sigma^2} \mu_{ML}}$$

* Fixed mean, latent variance

$$p(\underline{x}|\sigma^2) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2\right]$$

$$p(\sigma^2) = IG(\sigma^2; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}(\sigma^2)^{-(\alpha+1)} \exp\left(-\frac{\beta}{\sigma^2}\right)$$

$$p(\sigma^2|\underline{x}) \propto p(\underline{x}|\sigma^2)\, p(\sigma^2)$$

$$\propto (\sigma^2)^{-\frac{N}{2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2\right](\sigma^2)^{-(\alpha+1)} \exp\left(-\frac{\beta}{\sigma^2}\right)$$

$$\propto (\sigma^2)^{-(\alpha+1+\frac{N}{2})} \exp\left[-\frac{1}{\sigma^2}\left(\frac{1}{2}\sum_{i=1}^{N}(x_i-\mu)^2+\beta\right)\right]$$

On reconnaît la forme d'une IG :

$$p(\sigma^2|x) = IG(\sigma^2; \alpha_*, \beta_*) \propto (\sigma^2)^{-(\alpha_*+1)} \exp\left(-\frac{\beta_*}{\sigma^2}\right)$$

Par identification :

$$\boxed{\alpha_* = \alpha + \frac{N}{2}} \quad \text{et} \quad \boxed{\beta_* = \beta + \frac{1}{2}\sum_{i=1}^{N}(x_i-\mu)^2}$$